(Buffon's Needle Problem) A needle of length L is dropped randomly on a plane ruled with parallel lines that are distance D apart, where $D \geq L$. Show that the probability that the needle comes to rest crossing a line is $2L/(\pi D)$. Explain how this gives a mechanical means of estimating the value of π.

(sol.)

Let X be the distance from the center of the needle to the nearest line on the ruled plane. By the set up of the problem, it follows that $0 \leq X \leq \frac{D}{2}$ and because the needle can presumably fall anywhere on the plane with equal probability, we may assume that $X \sim$unif$([0, \frac{D}{2}])$. Let θ denote the acute angle (the angle less than $\frac{\pi}{2}$) between the needle and the nearest line. We may assume that $\theta \sim$unif$([0, \frac{\pi}{2}])$ since the needle may presumably land in any orientation with equal probability. Assuming further that the rotational orientation of the needle (θ) and the position of the needle relative to neighboring lines are independent, we know that the joint distribution of θ and X will be equal to the product to the marginal distributions. That is: $f_{X,\theta}(x, \theta) = (\frac{2}{D})(\frac{2}{\pi}) = \frac{4}{D\pi}$, and we'll use this to calculate the desired probability.

Then, the given a distance from the nearest line of X and an acute angle, θ, between the needle and the nearest line, we know that the end of the needle will just touch the line if $\sin(\theta) = \frac{x}{2/L}$ and that the needle will therefore cross the line whenever $\sin(\theta) \geq \frac{x}{2/L}$. Then, the desired probability is

$$P(X \geq \frac{L\sin(\theta)}{2}) = \int_0^{\frac{\pi}{2}} \int_0^{\frac{L}{2}\sin(\theta)} \frac{4}{D\pi} dx d\theta = \frac{4}{D\pi} \int_0^{\frac{\pi}{2}} \big[x\big]\Big|_0^{\frac{L}{2}\sin(\theta)} d\theta =$$

$$= \frac{2L}{D\pi} \int_0^{\frac{\pi}{2}} \sin(\theta) d\theta = \frac{2L}{D\pi} \big[-\cos(\theta)\big]\Big|_0^{\frac{\pi}{2}} = \frac{2L}{D\pi}$$

To estimate π, let the event that the needle crosses a line be denoted by A. Then:

$$P(A) = \frac{2L}{D\pi} \implies \pi = \frac{2L}{D\, P(A)} = \frac{2L}{D} \frac{1}{P(A)}$$

Suppose we drop n needles and record the proportion of times the needle crosses a line; let us denote this by $P(A)_n$. Clearly, $P(A)_n \to P(A)$ as $n \to \infty$ and we see that:

$$\lim_{n \to \infty} \frac{2L}{D} \frac{1}{P(A)_n} = \frac{2L}{D} \frac{1}{P(A)} = \pi$$

So, we may mechanically approximate π by setting $P(A)_n = \frac{t}{n}$ in the above expression, where t/n is the ratio of times that the needle crosses a line in n trials and we see that

$$\frac{2L}{D} \frac{n}{t} \to \pi \quad \text{as} \quad \frac{n}{t} \to \frac{1}{P(A)}$$